A Bayesian modeling approach to fuzzy data analysis

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Introduction

Social surveys are widespread tools used to collect data and insights on people's attitudes, behaviors, and characteristics within a population.

Rating data are a common method for representing information from social surveys, allowing the quantification of opinions and experiences.

Although simple and effective, rating data can introduce uncertainty by not capturing the **complexity of respondents' opinions**.

Introduction

The complexity arises due to the **interplay** of cognitive, affective and contextual factors in the **process of answering** questions using rating scales.

Hence, rating data encapsulate both the rater's **final response** and **epistemic uncertainty**. This type of **post-sampling** uncertainty also coexists with the uncertainty induced by the sampling process.

Fuzzy numbers can be used to mathematically represent this source of uncertainty as **epistemic imprecision** (or **fuzziness**).

Introduction

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Introduction

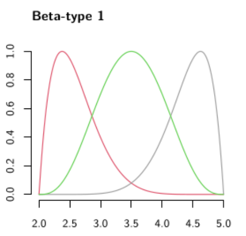
To deal with fuzziness and randomness appropriately, we need to:

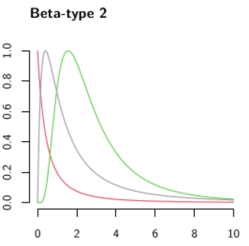
⇒ **Generalize** the statistical modeling to accommodate both sources of uncertainty simultaneously

⇒ Consider that **estimators** could suffer from **excessive variance** when **epistemic fuzzy data** are used [Grzegorzewski and Goławska, 2021]

⇒ Develop a **general** and consistent statistical modeling framework to deal with **fuzzy data analysis**

Introduction





Beta-type fuzzy numbers as a general template for representing **continuous** and **unimodal** fuzzy numbers:

- flexible and parsimonious as they require two parameters only $\{m, s\} \in [lb, ub] \times \mathbb{R}^+$ (mode and precision)
- allow for dealing with variables supported on bounded or semi-infinite intervals (as usual in socio-economic research)
- generalize frequently used fuzzy numbers (triangular, trapezoidal)

Statement of the problem

Let $Y_1, \ldots, Y_n$ be n independent continuous RVs and $\tilde{\mathbf{y}} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ a sample of fuzzy observations. The vector $\tilde{\boldsymbol{y}}$ is a **blurred** realization of $\mathbf{y}$ because of *post-sampling* or *epistemic* uncertainty-based processes.

$\boxed{\text{The interest lies in studying } f_{Y_1,\ldots,Y_n}(\mathbf{y}; \boldsymbol{\theta}_{\boldsymbol{y}}) \text{ with the purpose of making inference on } \boldsymbol{\theta}_{\boldsymbol{y}} \text{ given a fuzzy sample } \tilde{\boldsymbol{y}}.}$

Each fuzzy observation $\tilde{y}_i$ consists of its mode and precision $\{m_i, s_i\}$ of a Beta-type fuzzy number.

A conditional sampling schema

The idea is to use a **conditional schema** linking the statistics of fuzzy numbers to $f_{Y_1,\ldots,Y_n}(\mathbf{y};\boldsymbol{\theta}_\mathbf{y})$:

$$y_i \sim f_Y(y;\boldsymbol{\theta}_\mathbf{y})$$

$$s_i \sim f_S(s;\boldsymbol{\theta}_\mathbf{s})$$

$$m_i|y_i,s_i \sim f_{M|S,Y}(m;\omega(y,s))$$

A conditional sampling schema

$$
\begin{aligned}
\boxed{y_i \sim f_Y(y; \boldsymbol{\theta}_{\boldsymbol{y}})} \\
s_i \sim f_S(s; \boldsymbol{\theta}_{\boldsymbol{s}}) \\
m_i | y_i, s_i \sim f_{M|S,Y}(m; \omega(y, s))
\end{aligned}
$$

RV that governs the stochastic (**non-fuzzy**) sampling process. The parameters can be expressed as a function of external covariates $\boldsymbol{\theta}_{\boldsymbol{y}} = g^{-1}(\mathbf{X}\beta)$ as for GLMs.

The choice of $f_Y(y; \boldsymbol{\theta}_{\boldsymbol{y}})$ depends on the specific problem one is dealing with (e.g., Beta distribution, Logistic distribution, Weibull distribution).

A conditional sampling schema

$$y_i \sim f_Y(y; \boldsymbol{\theta_y})$$
$$\boxed{s_i \sim \mathcal{G}a(s; \alpha_s, \beta_s)}$$
$$m_i | y_i, s_i \sim f_{M|S,Y}(m; \omega(y, s))$$

Gamma distribution with $\alpha_s > 0$ and $\beta_s > 0$ modeling the precision (or spread) of the fuzzy number. In the simplest case, $s_i \perp\!\!\!\perp y_i$ although it can be generalized to cope with cases where s_i depends on y_i or external covariates.

A conditional sampling schema

$$y_i \sim f_Y(y; \boldsymbol{\theta_y})$$
$$s_i \sim \mathcal{G}a(s; \alpha_s, \beta_s)$$
$$\boxed{m_i|y_i, s_i \sim f_{M|S,Y}(m; \omega(y, s))}$$

RV for the mode of the fuzzy number as a function of the true unobserved outcome y_i and the spread s_i.

A conditional sampling schema

$$
\begin{aligned}
y_i &\sim f_Y(y; \boldsymbol{\theta_y}) \\
s_i &\sim \mathcal{G}a(s; \alpha_s, \beta_s) \\
&\boxed{m_i | y_i, s_i \sim f_{M|S,Y}(m; \omega(y,s))}
\end{aligned}
$$

Case 1: $y \in (lb, ub)$, $f_{M|S,Y}(m; \omega(y,s))$ is the **4-parameter Beta distribution**

Case 2: $y \in (0, +\infty)$, $f_{M|S,Y}(m; \omega(y,s))$ is the **Beta prime distribution**

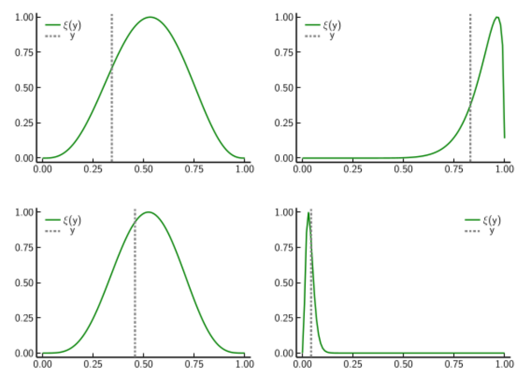
A conditional sampling schema

$$y_i \sim f_Y(y; \boldsymbol{\theta_y}) \tag{1}$$

$$s_i \sim \mathcal{G}a(s; \alpha_s, \beta_s) \tag{2}$$

$$m_i | s_i, y_i \sim \begin{cases} \mathcal{B}e_{4P}(m; s_i y_i, s_i - s_i y_1, lb, ub), & \text{if } y_i \in (lb, ub) \\ \mathcal{B}e_P(m; y_i + y_i s_i, s_i + 2), & \text{if } y_i \in (0, +\infty) \end{cases} \tag{3}$$

A conditional sampling schema



Examples of a Beta-type 1 fuzzy number $\xi_{\widetilde{y}}$ masking the (true) uncorrupted realizations y

Inference on $\boldsymbol{\theta}_y$

Inference about $\boldsymbol{\theta}_y$ involves a kind of **deblurring** procedure which uses $\tilde{\boldsymbol{y}}$ instead of the unobserved realizations $\mathbf{y}$.

Inference on $\boldsymbol{\theta}_y$

The idea is to plug the hypothesized sampling schema into the estimation procedure, which naturally leads to the **Gibbs sampler**-based solution:

For $t > 1$ do:

$$\mathbf{y}^{(t)} \sim \pi(\mathbf{y}|\mathbf{m}, \mathbf{s}, \boldsymbol{\theta}_y^{(t-1)})$$

$$\boldsymbol{\theta}_{\boldsymbol{y}}^{(t)} \sim \pi(\boldsymbol{\theta}_{\boldsymbol{y}}|\mathbf{m}, \mathbf{s}, \boldsymbol{y}^{(t)})$$

For large T inference on $\boldsymbol{\theta}_{\boldsymbol{y}}$ can be performed by an inspection of the posterior sequence $\left(\boldsymbol{\theta}_{\boldsymbol{y}}^{(1)}, \ldots, \boldsymbol{\theta}_{\boldsymbol{y}}^{(T)}\right)$.

Inference on $\boldsymbol{\theta}_y$

Conditional posterior densities $\pi(\mathbf{y}| \ldots)$ and $\pi(\boldsymbol{\theta}_{\mathbf{y}}| \ldots)$ have unknown forms under the proposed sampling schema. Then, hybrid solutions, such as **posterior approximation** or the **Metropolis within Gibbs** could be used to solve the problem.

Inference on $\boldsymbol{\theta}_y$

<u>Posterior sampling schema</u>

$$\pi(\mathbf{y}, \boldsymbol{\theta}_{\boldsymbol{y}} | \{\mathbf{m}, \mathbf{s}\}) \quad \begin{array}{l} \nearrow \quad \pi(\mathbf{y}|\boldsymbol{\theta}_{\boldsymbol{y}}, \{\mathbf{m}, \mathbf{s}\}) \quad \text{via a } \textbf{posterior approximation} \\[2em] \searrow \quad \pi(\boldsymbol{\theta}_{\boldsymbol{y}}|\mathbf{y}, \{\mathbf{m}, \mathbf{s}\}) \quad \text{via } \textbf{MCMC} \end{array}$$

Inference on $\boldsymbol{\theta}_y$

Posterior sampling schema

$$\pi(\mathbf{y}|\boldsymbol{\theta}_{\mathbf{y}}, \{\mathbf{m}, \mathbf{s}\}) \quad \text{via a } \textbf{posterior approximation}$$

$$\stackrel{\sim}{=} \mathcal{B}e_{4P}(y; \lambda\sigma, \sigma - \sigma\lambda, lb, ub) \qquad \text{(case 1)}$$

$$\stackrel{\sim}{=} \mathcal{B}e_{P}(y; \lambda + \lambda\sigma, \sigma + 2) \qquad \text{(case 2)}$$

Inference on $\boldsymbol{\theta}_y$

Posterior sampling schema

$\pi(\mathbf{y}|\boldsymbol{\theta}_{\boldsymbol{y}}, \{\mathbf{m}, \mathbf{s}\})$ via a **posterior approximation**

$$\stackrel{\sim}{=} \mathcal{B}e_{4P}(y; \lambda\sigma, \sigma - \sigma\lambda, lb, ub) \qquad \text{(case 1)}$$

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$\{\lambda, \sigma\}$ found by derivative matching [Miller, 2019]

Inference on $\boldsymbol{\theta}_y$

Posterior sampling schema

$\pi(\boldsymbol{\theta}_{\boldsymbol{y}}|\mathbf{y},\{\mathbf{m},\mathbf{s}\})$ via **MCMC**

using the Vihola's Robust Adaptive MH algorithm with a coerced acceptance rate [Vihola, 2012]

Inference on $\boldsymbol{\theta}_y$

Simulation studies show the effectiveness of the approximated hybrid Gibbs sampling to accurately estimate model parameters with a good mixing properties.

Inference on $\boldsymbol{\theta}_y$

Simulation studies show the effectiveness of the approximated hybrid Gibbs sampling to accurately estimating model parameters with a good mixing property.

$\Rightarrow$ Instead, in the next slides we focus on the ability of the proposed conditional schema to reproduce already existing fuzzy data (**external validation**).

Case studies

Aim: Assessing the capability of the proposed conditional schema to reproduce fuzzy data $\tilde{\mathbf{y}}$ collected externally.

Method: Posterior predictive check [Gelman et al., 1996]

i) Fix the non-fuzzy model $f_Y(y; \boldsymbol{\theta})$

ii) Estimate $\hat{\boldsymbol{\theta}}$ using $\tilde{\mathbf{y}}$

iii) Generate B new instances $\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_B$ using the conditional schema

Measures: Compare observed statistics $S(\tilde{\mathbf{y}})$ - i.e., *centroids*, 0-*cuts*, *fuzziness* - with the distribution of the simulated ones through the range and Q_3-Q_1 interquartile range.

Case studies

Dataset 1

Dataset: Sample of $n = 69$ observations about Reckless Driving Behavior collected using fuzzy indirect rating scales [Calcagnì and Lombardi, 2022].

Response variable: Driving Anger Scale (DAS) represented as Beta type-1 fuzzy numbers.

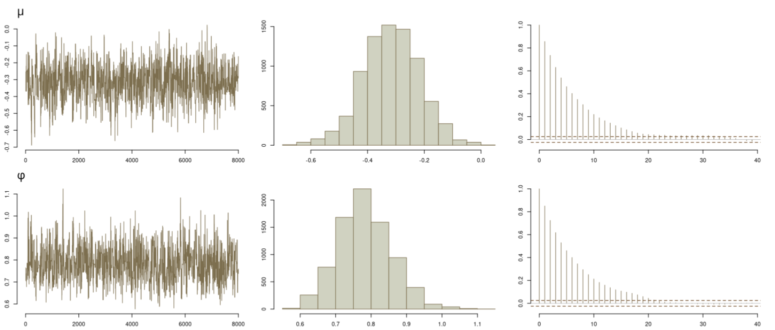
Non-fuzzy model: $f_Y(y;\boldsymbol{\theta}) = \mathcal{L}\text{ogit}\mathcal{N}\text{orm}(y;\mu,\phi)$ with $\{\mu,\sigma\} \in \mathbb{R} \times \mathbb{R}^+$.

Covariates: None.

MCMC: $f(\mu) = \mathcal{N}(;0,100)$, $f(\sigma) = \mathcal{U}(;0,100)$; $\boldsymbol{\theta}_s$ estimated via ML;
No. of samples $1e4$, Burn-in $2.5e3$, acc. rate 0.3045; No. of predictions $B = 5e2$.

Case studies

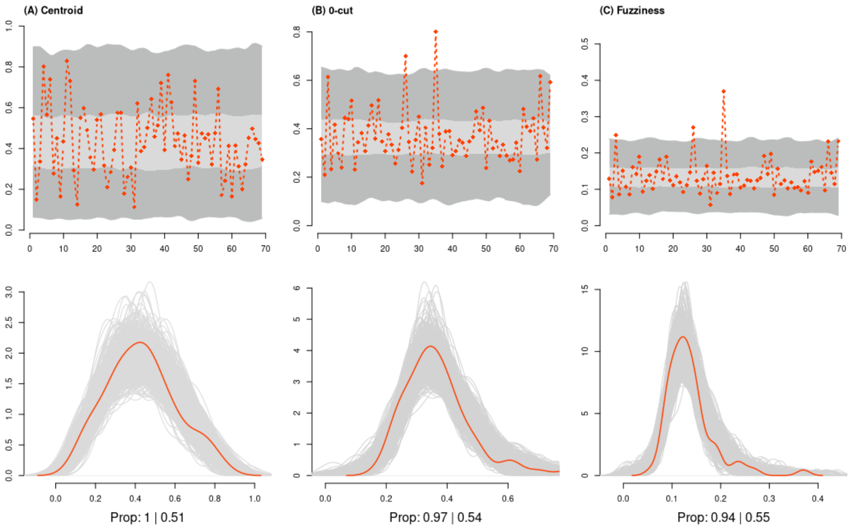
Dataset 1



	Q_1	Median	Mean	Q_3
μ	-0.38	-0.32	-0.32	-0.25
σ	0.73	0.78	0.78	0.83

Case studies

Dataset 1



Case studies

Dataset 2

Dataset: Sample of $n = 49$ observations about Restaurant Quality collected using direct fuzzy rating scales [de Sáa et al., 2014].

Response variable: Restaurant Quality (QR7) represented as Beta type-1 fuzzy numbers (converted from triangular/trapezoidal).

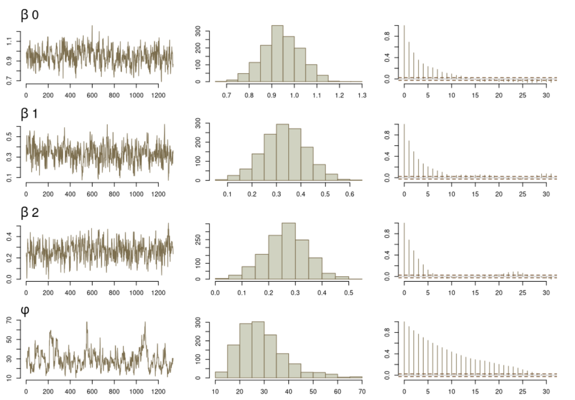
Non-fuzzy model: $f_Y(y;\boldsymbol{\theta}) = Be(y;\boldsymbol{\mu},\phi)$ with $\{\boldsymbol{\mu},\phi\} \in [0,1]^n \times \mathbb{R}^+$
$\mu_i = \text{logit}^{-1}(\mathbf{x}_i\boldsymbol{\beta})$.

Covariates: `quality_food`, `quality_employees` (composite indicators from crisp variables).

MCMC: $f(\beta_j) = \mathcal{N}(;0,100)$ $(j = 1,\ldots,4)$, $f(\phi) = \mathcal{U}(;0,100)$; $\boldsymbol{\theta}_s$ estimated via ML;
No. of samples $1e4$, Burn-in $2.5e3$, acc. rate 0.2554; No. of predictions $B = 5e2$.

Case studies

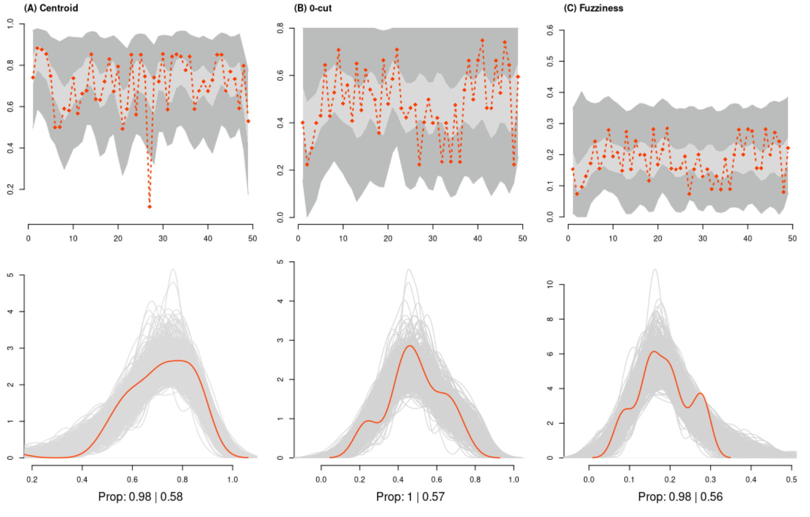
Dataset 2



	Q_1	Median	Mean	Q_3
β_0	0.89	0.94	0.94	1.00
β_1	0.28	0.33	0.33	0.39
β_2	0.21	0.27	0.27	0.32
ϕ	22.36	27.40	29.15	33.92

Case studies

Dataset 2



Case studies

Dataset 3

Dataset: Sample of $n = 147$ observations about Shangai's House Prices collected using fuzzy conversion scales [Zhou et al., 2018].

Response variable: Purchase price represented as Beta type-2 fuzzy numbers (converted from triangular).

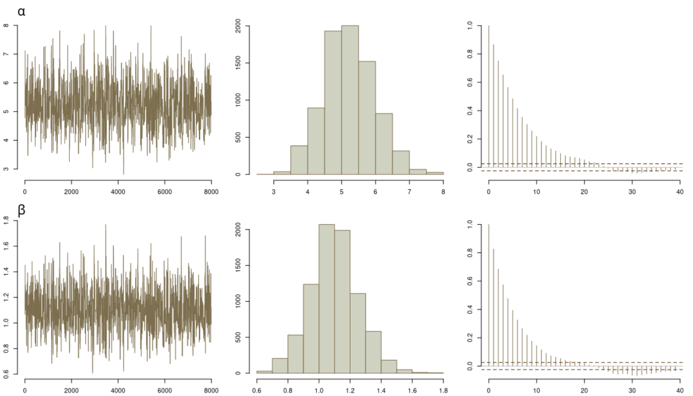
Non-fuzzy model: $f_Y(y;\boldsymbol{\theta}) = Ga(y;\boldsymbol{\alpha},\beta)$ with $\{\alpha,\beta\} \in \mathbb{R}^+ \times \mathbb{R}^+$

Covariates: None.

MCMC: $f(\alpha) = \mathcal{U}(;0,100)$, $f(\beta) = \mathcal{U}(;0,100)$; $\boldsymbol{\theta}_s$ estimated via ML;
No. of samples $1e4$, Burn-in $2.5e3$, acc. rate 0.2876; No. of predictions $B = 5e2$.

Case studies

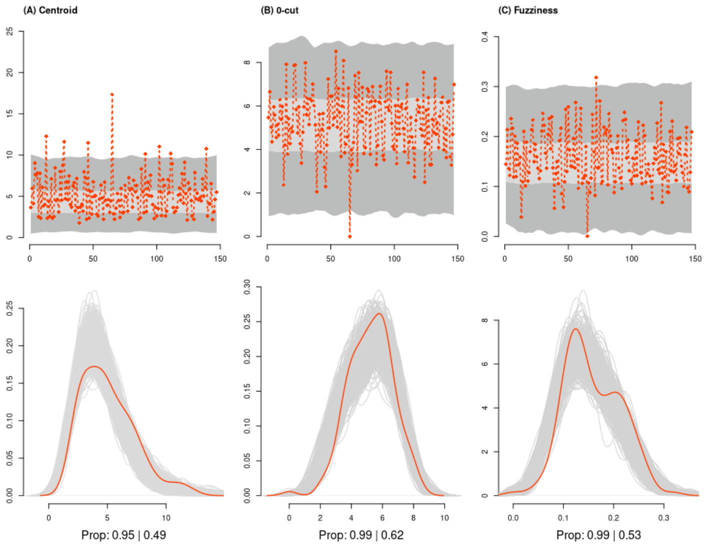
Dataset 3



	Q_1	Median	Mean	Q_3
α	4.70	5.20	5.21	5.69
β	1.00	1.10	1.10	1.19

Case studies

Dataset 3



Concluding Remarks

- Statistical modeling with fuzzy numbers can be of relevant importance in all those situations involving non-stochastic sources of uncertainty (e.g., decision uncertainty in answering a social survey)
- A general and consistent statistical modeling framework to deal with fuzzy data analysis is necessary for practitioners (GLMs-like approach)

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- A general and consistent statistical modeling framework to deal with fuzzy data analysis is necessary for practitioners (GLMs-like approach)
- The proposed schema is entirely probabilistically: fuzziness is summarized into (a few) statistics (e.g., FDA, Network Data Analysis)
- The assumption $s_i \perp\!\!\!\perp y_i$ can be unrealistic in many circumstances

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