Integrating Rasch and Compositional modeling for the analysis of social survey data

Antonio Calcagnì
University of Padova, Italy

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Introduction

Rating data capture human attitudes and opinions but may hold more information than typically conveyed.

Traditional questionnaires capture the final responses, missing insights into individuals' **decision-making processes**, which could reveal variations in hesitancy and uncertainty.

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Traditional questionnaires capture the final responses, missing insights into individuals' **decision-making processes**, which could reveal variations in hesitancy and uncertainty.

Due to the **interplay** of cognitive, affective and contextual factors in the **process of answering** multiple choice tasks, rating data can disclose more information if appropriately handled.

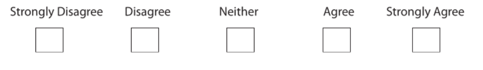
Introduction

Rating as decision-making process

To illustrate this point, consider the question

I am satisfied with my current work

alongside a five-point scale:



Introduction

Rating as decision-making process

To answer,

$\Rightarrow$ the rater retrieves long-term memory information about events and beliefs about their job

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Rating as decision-making process

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⇒ the rater retrieves long-term memory information about events, beliefs about their job

⇒ these information activate affective components, which influence positively or negatively the **opinion formation** (for example, a recent promotion may enhance the chance for answering the item positively)

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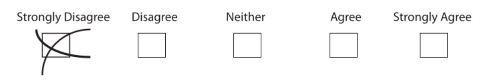
Rating as decision-making process

To answer

⇒ the rater retrieves long-term memory information about events, beliefs about their job

⇒ these information activate affective components which influence positively or negatively the opinion formation (for example, a recent promotion may enhance the chance for answering the item positively)

⇒ cognitive and affective information are integrated to activate the decision making stage and provide the **final response**



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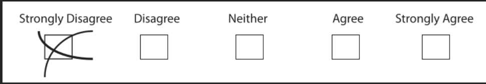
Rating as decision-making process



Due to conflicting demands in these stages, **decision uncertainty** at certain levels can arise and influence the rating decision.

As a result, the final response does not tell the whole story.

Introduction

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⇒ via Item Response Theory: Binary tree-based Rasch model [1]

⇒ via Compositional Data Analysis: Dirichlet regression [2, 4]

Methods

Rasch-IRTree data representation

The **Rasch-tree model** is a member of the IRTrees family [1] and provides a statistical representation of rating responses using **conditional binary trees**.

Methods

Rasch-IRTree data representation

To describe how it works, consider again the previous example:

- I am satisfied with my current work -

| Strongly disagree | — | Disagree | — | 0 | — | Agree | — | Strongly agree |

Methods

Rasch-IRTree data representation

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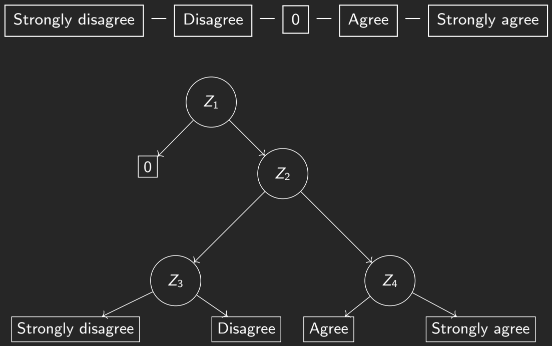
| Strongly disagree | — | Disagree | — | 0 | — | Agree | — | Strongly agree |

Then, each **response option** is thought as being the output of a cognitive sub-process of the entire response process. The sub-processes are modeled as **nodes** of a **binary tree**.

Methods

Rasch-IRTree data representation

An example of 5-point rating scale with the associated binary decision tree.

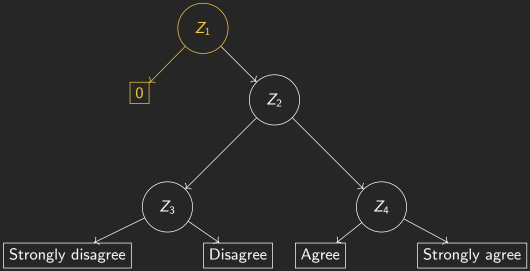


Methods

Rasch-IRTree data representation

In this schema, the rater:

- first decides **whether or not** provide a response ($Z_1 \in \{0, 1\}$)



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In this schema, the rater:

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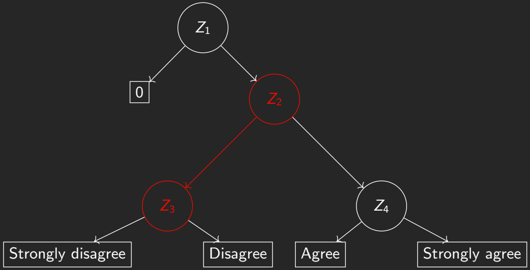
- then, for $Z_1 = 1$ he/she decides the **direction** of the response, if negative ($Z_2 = 0$)



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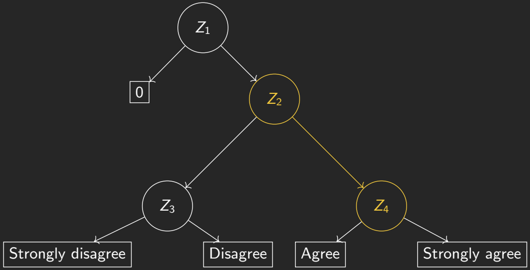

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Rasch-IRTree data representation

In this schema, the rater:

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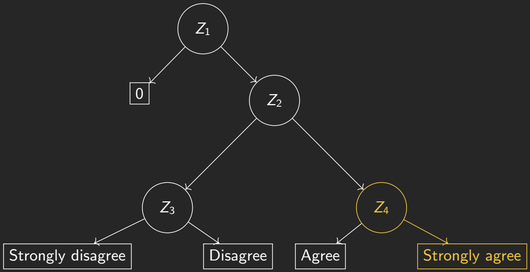
- then, for $Z_1 = 1$ he/she decides the direction of the response, if negative ($Z_2 = 0$) or **positive** ($Z_2 = 1$)
- finally, he/she decides the **strength** of the response, e.g. "Strongly agree" ($Z_4 = 1$)



Methods

Rasch-IRTree data representation

The Rasch-tree model is defined by the following equations:
(i-th rater, j-th item, n-th node)

$$Z_{ijn} \sim \mathcal{B}er(\pi_{ijn})$$

$$\pi_{ijn} = \mathbb{P}(Z_{in} = 1; \boldsymbol{\theta}_n) = \frac{\exp(\eta_{in} + \alpha_{jn})}{1 + \exp(\eta_{in} + \alpha_{jn})}$$

$$\eta_{in} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

where

$\alpha_{jn} \in \mathbb{R}$: easiness of the **item** being rated

$\eta_{in} \in \mathbb{R}$: **rater**'s latent ability to answer the question

Methods

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where

$$\mathbb{P}(Y_i = m; \boldsymbol{\theta}_n) = \prod_{n=1}^{N} \mathbb{P}(Z_{in} = d; \boldsymbol{\theta}_n)^d$$

is the probability of the response $Y_i = m$ for the item being rated.

Methods

Rasch-IRTree data representation

The Rasch-tree model is defined by the following equations:
(*i*-th rater, *j*-th item, *n*-th node)

$$
\begin{aligned}
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& \eta_{in} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\eta)
\end{aligned}
$$

The parameters $\boldsymbol{\theta}_n = \{\boldsymbol{\alpha}, \boldsymbol{\Sigma}_\eta\}$ can be estimated via **marginal maximum likelihood** [1].

Methods

Rasch-IRTree data representation

Once $\hat{\alpha}$ and $\widehat{\boldsymbol{\Sigma}}_\eta$ have been recovered conditioned on a sample of data $\mathbf{Y}_{I\times J}$, the estimated **transition probabilities**

$$\mathcal{U}_i = \Big(\hat{\mathbb{P}}(Y_i = 1), \ldots, \hat{\mathbb{P}}(Y_i = m), \ldots, \hat{\mathbb{P}}(Y_i = M)\Big)$$

provide information about the inner mechanisms of the rater's response process.

Methods

Rasch-IRTree data representation

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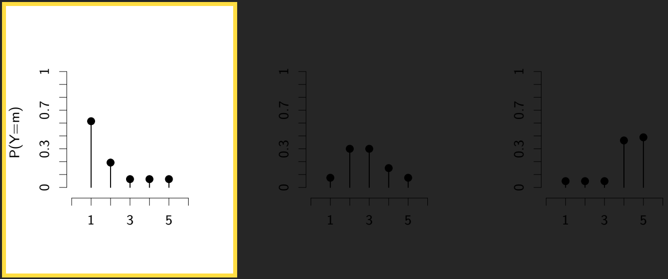

Response process with lower degree of decision uncertainty
(i.e., the response $Y = 1$ is more certain than the remaining ones).

Methods

Rasch-IRTree data representation

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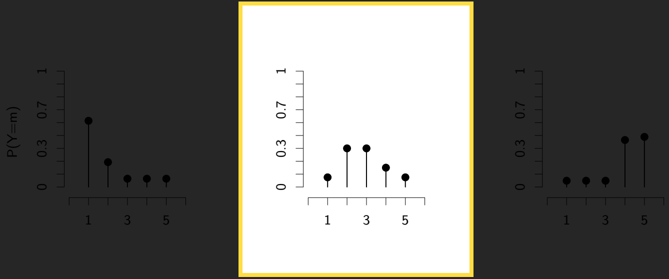
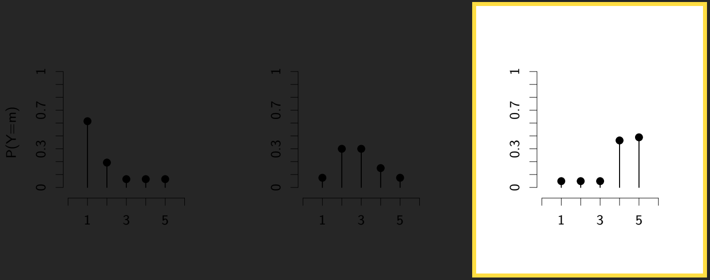


Response process with higher degree of decision uncertainty (i.e., both $Y \in \{2, 3\}$ responses are probable).

Methods

Rasch-IRTree data representation

$$\mathcal{U}_i = \Big(\hat{\mathbb{P}}(Y_i = 1), \ldots, \hat{\mathbb{P}}(Y_i = m), \ldots, \hat{\mathbb{P}}(Y_i = M)\Big)$$



Response process with a certain degree of decision uncertainty (i.e., both $Y \in \{4, 5\}$ responses are probable).

Methods

Rasch-IRTree data representation

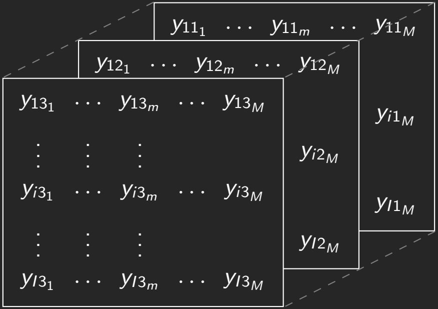
The IRTree procedure can be seen as a form of scale quantification, which outputs a set of M probability masses for each respondent i and survey item j.

$$\mathbf{Y}_{I\times J} \rightarrow \boxed{\text{Rasch-tree}} \rightarrow \widetilde{\mathbf{Y}}_{I\times J\times M}$$

Methods

Rasch-IRTree data representation

$$\mathbf{Y}_{n\times J} \rightarrow \boxed{\text{Rasch-tree}} \rightarrow \widetilde{\mathbf{Y}}_{n\times J\times M}$$



$$y_{ij} \in \{1,\ldots,M\}$$

$$\tilde{\mathbf{y}}_{ij} \in \left\{\mathbf{y} \in \mathbb{R}_+^M : \mathbf{1}^T\mathbf{y} = 1\right\}$$

$$\bar{\mathbf{y}}_{ij} \in [0,M] \subset \mathbb{R} \quad \text{(expectation)}$$

$\Rightarrow$ quantified response

Methods

Dirichlet compositional regression

Let $\widetilde{\mathbf{Y}} = (\widetilde{\mathbf{y}}_1, \ldots, \widetilde{\mathbf{y}}_i, \ldots, \widetilde{\mathbf{y}}_n)$ be a collection of independent random compositions with

$$\tilde{\mathbf{y}}_i \in \mathbb{S}^M, \quad \mathbb{S}^M = \left\{ (y_1, \ldots, y_m, \ldots, y_M) \in \mathbb{R}_+^M : \mathbf{1}^T \mathbf{y} = 1 \right\}$$

Let $\mathbf{X}_{n \times J}$ be a matrix of observed variables (e.g., covariates).

The Dirichlet linear model with fixed dispersion ϕ is:

$$\tilde{\mathbf{y}}_i \sim \mathcal{D}(y; \boldsymbol{\mu}_i \phi), \quad g(\boldsymbol{\mu}_i) = \mathbf{x}_i \boldsymbol{\beta}$$

$$\boldsymbol{\mu}_i = \left(\frac{1}{\sum_{m=1}^M \exp \mathbf{x}_i \beta_m}, \frac{\exp \mathbf{x}_i \boldsymbol{\beta}_2}{\sum_{m=1}^M \exp \mathbf{x}_i \beta_m}, \ldots, \frac{\exp \mathbf{x}_i \boldsymbol{\beta}_M}{\sum_{m=1}^M \exp \mathbf{x}_i \beta_m} \right)$$

with $\boldsymbol{\beta}_1 = \mathbf{0}_J$ being the reference level. The parameters estimation is performed via maximum likelihood [4].

Case study

Data description

Aim: Investigate the predictors of anxiety in watching the war in Ukraine [3].

Sample: $n = 796$ respondents from Canada, Germany, and Finland

- 68% female, mean age 24.4 years
- 85% did not have relatives or friends involved in the war

Variables:

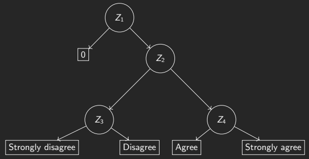
- Response: `anxiety` measured using a six-item questionnaire (with $M = 5$)
- Predictors: `gender` and `depression` (total score from a eight-item questionnaire)

Case study

Data analysis and results

On the first 50% of the dataset ($n = 398$):

- The simple linear decision tree used for modeling `anxiety`
- A random-effect Binomial linear model used to estimate $\hat{\boldsymbol{\alpha}}_{6\times(M-1)}$ for each of the six items and $\hat{\boldsymbol{\Sigma}}_{\eta_{(M-1)\times(M-1)}}$

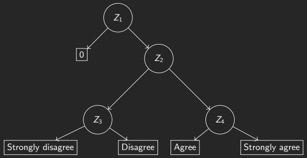


Case study

Data analysis and results

On the second 50% of the dataset ($n = 398$):

- $\hat{\mathbb{P}}(Y_{ij} = 1), \ldots, \hat{\mathbb{P}}(Y_{ij} = M)$ computed for all the items and respondents



Case study

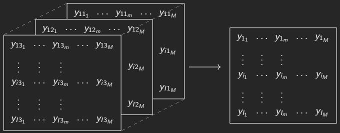
Data analysis and results

On the second 50% of the dataset ($n = 398$):

- $\hat{\mathbb{P}}(Y_{ij} = 1), \ldots, \hat{\mathbb{P}}(Y_{ij} = M)$ computed for all the items and respondents (i.e., $\tilde{\mathbf{Y}}_{I \times J \times M}$)

- According to the Aitchinson's geometry, the **compositional total score** of `anxiety` was computed using the perturbation average (i.e., $\tilde{\mathbf{Y}}_{I \times M}$):

$$\tilde{\mathbf{y}}_i = \frac{1}{J} \odot \bigoplus_{j=1}^{J} \mathbf{y}_{ij}$$



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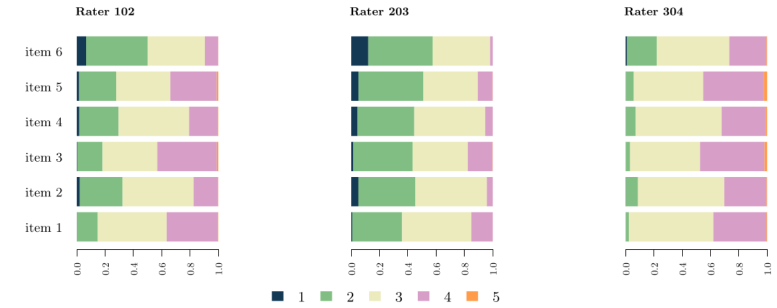


Compositional responses for a selection of respondents

Case study

Data analysis and results

On the second 50% of the dataset ($n = 398$):

- Dirichlet linear model with logit (μ) and log (ϕ) links
- μ included `depression`, `gender`, and `depression`$\times$`gender`

Case study

Data analysis and results

On the second 50% of the dataset ($n = 398$):

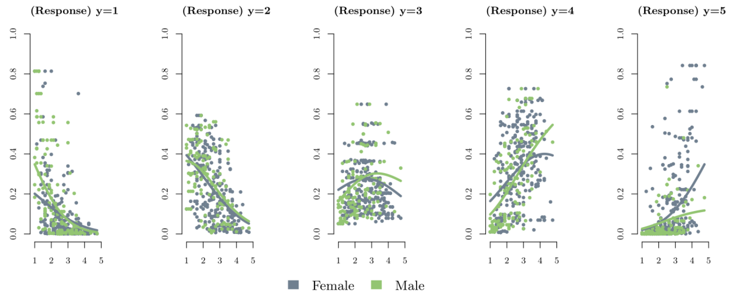
- Dirichlet linear model with logit (μ) and log (ϕ) links
- μ included `depression`, `gender`, and `depression`$\times$`gender`
- $Y = 1$: *"Not at all"* reference level for μ

	$Y=2$			$Y=3$		
	$\hat{\theta}$	$\sigma_{\hat{\theta}}$	z	$\hat{\theta}$	$\sigma_{\hat{\theta}}$	z
β_0	0.57	0.25	2.28	-0.51	0.24	-2.08
β_{depres}	0.10	0.09	1.11	0.60	0.09	6.74
$\beta_{\text{gender:Male}}$	-1.04	0.37	-2.84	-1.36	0.37	-3.71
$\beta_{\text{depres}\times\text{gender}}$	0.41	0.15	2.69	0.51	0.15	3.41

	$Y=4$			$Y=5$		
	$\hat{\theta}$	$\sigma_{\hat{\theta}}$	z	$\hat{\theta}$	$\sigma_{\hat{\theta}}$	z
β_0	-1.08	0.24	-4.54	-3.70	0.29	-12.87
β_{depres}	0.87	0.09	10.13	1.40	0.10	13.86
$\beta_{\text{gender:Male}}$	-1.64	0.36	-4.49	-0.27	0.43	-0.64
$\beta_{\text{depres}\times\text{gender}}$	0.57	0.15	3.85	-0.02	0.17	-0.11
ϕ	1.81	0.03	53.03			

Case study

Data analysis and results

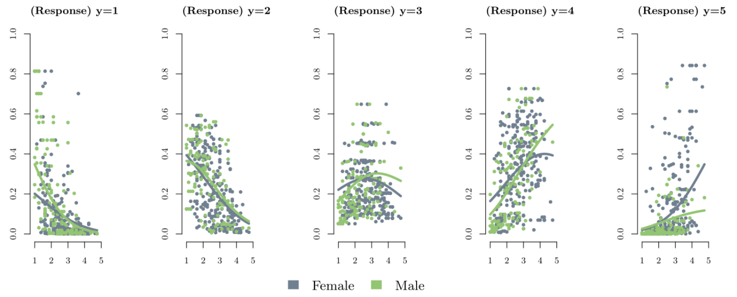


As depression increases, respondents are more likely to self-report substantial anxiety compared to the baseline *"Not at all"*.

The odds of experiencing extreme anxiety ($Y = 5$) increase by a factor of $\exp(\beta_{\text{depres}}) = 4.05$ compared to having no anxiety at all.

Case study

Data analysis and results



The interaction depress×gender reveals that, compared to females with no anxiety, males are approximately 1.70 times more likely to experience moderate ($Y = 3$) or high ($Y = 4$) levels of anxiety when experiencing depression.

Conclusions

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□ The IRtree statistical model acts as a (kind of) denoiser of the observed data.

Conclusions

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  Cons: It requires large dataset to estimate α and Σ appropriately.

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- It paves the way for analysing data using more informative statistical techniques (i.e., COmpositional Data Analysis).

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- The IRtree statistical model acts as a (kind of) denoiser of the observed data.

  Cons: It requires large dataset to estimate $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$ appropriately.

- It paves the way for analysing data using more informative statistical techniques (i.e., COmpositional Data Analysis).

  Cons: Sometimes, it lacks a clear and immediate interpretation of the results.

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antonio.calcagni@unipd.it

https://unipd.link/acalcagni