A Bayesian beta linear model for fuzzy rating responses

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- *I am satisfied with my life* -

using the graded scale:

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Rating data are common when measuring human-based characteristics where attitudes, motivations, satisfaction, or beliefs are quantified using rating scales.

A typical example is that of rating the question:

- *I am satisfied with my current work* -

using the graded scale:

Strongly disagree — Disagree — 0 — Agree — Strongly agree

As they involve human raters, rating data are often affected by **fuzziness** because of the **decision uncertainty** that affects the **response process**.

Several methods might be adopted to quantify fuzziness (**fuzzy scaling**):

- direct fuzzy rating [3]
- implicit fuzzy rating [1]
- deterministic crisp-to-fuzzy conversion systems [7]
- statistically-oriented crisp-to-fuzzy conversion systems [8]

Besides their differences, all these approaches aim at quantifying the fuzziness present in rating data.

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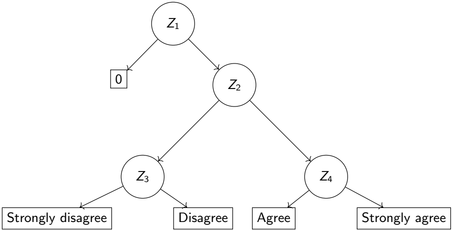
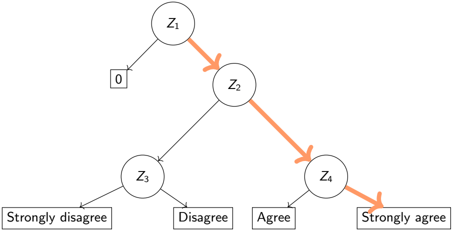
Key idea: The entire response process can be modeled stage-wise by means of an *Item Response Theory tree*:

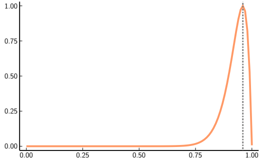
Binary trees + Rasch psychometric model

IRTree-based fuzzy numbers



Fuzziness arises as a result of the **transitions probabilities** estimated by the IRTree (i.e., *the easier the transition, the lesser the fuzziness*).





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An example of rating response with **low degree of fuzziness**.

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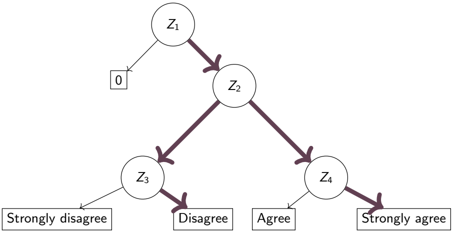

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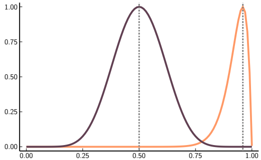

An example of rating response with **low degree of fuzziness**.

An example of rating response with **high degree of fuzziness**.

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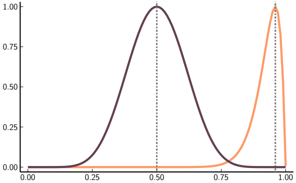

Fuzzy numbers are derived from the **estimated parameters** of the IRTree model. Further details in [2].

Several choices are available to this purpose. **Beta fuzzy numbers** are a good candidate in terms of flexibility and simplicity:

$$\xi_{\widetilde{y}_{ij}}(y) = \frac{1}{C}\; y^{m_{ij}s_{ij}}(1-y)^{s_{ij}-s_{ij}m_{ij}} \qquad m_{ij} \in (0,1) \quad s_{ij} \in \mathbb{R}^+$$

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For a non-fuzzy collection of i.i.d. $(0,1)$-realizations $\mathbf{y} = (y_1, \ldots, y_n)$, the **Beta density** is as follows:

$$f_{\mathbf{Y}}(\mathbf{y}; \boldsymbol{\mu}, \boldsymbol{\phi}) = \prod_{i=1}^{n} \frac{\Gamma(\phi_i)}{\Gamma(\phi_i\mu_i)\Gamma(\phi_i - \mu_i\phi_i)}\; y_i^{(\mu_i\phi_i - 1)}(1-y_i)^{(\phi_i - \mu_i\phi_i - 1)}$$

where

$$\boldsymbol{\mu} = (1+\exp(\mathbf{X}\boldsymbol{\beta}))^{-1} \quad \text{and} \quad \boldsymbol{\phi} = \exp(\mathbf{Z}\boldsymbol{\gamma})$$

where $\boldsymbol{\mu} \in (0,1)^n$ is the $n \times 1$ vector of location parameters and $\boldsymbol{\phi} \in (0,\infty)^n$ the $n \times 1$ vector of precision parameters, which have been linearly expanded to account for covariates $\mathbf{X}_{n\times p}$ and $\mathbf{Z}_{n\times q}$.

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Model parameters: $\boldsymbol{\theta} \in \{\boldsymbol{\beta}, \boldsymbol{\gamma}\} \in \mathbb{R}^p \times \mathbb{R}^q$

To sample from the posterior density of the model parameters $f(\boldsymbol{\beta}, \boldsymbol{\gamma})$, an **adaptive Metropolis-Hastings** algorithm has been used [5], with the **transition densities** of the MCMCs being defined as:

$$q(\boldsymbol{\theta}^{(t)}|\boldsymbol{\theta}^{(t-1)}) = \mathcal{N}(;\boldsymbol{\theta}^{(t-1)}, \boldsymbol{\Sigma}^{(t)})$$

The acceptance ratio of the sampler is:

$$\alpha^{(t)} = \frac{\tilde{\mathcal{L}}(\boldsymbol{\theta}^{(t)}; \mathbf{m}, \mathbf{s})\ q(\boldsymbol{\theta}^{(t-1)}|\boldsymbol{\theta}^{(t)})\ f(\boldsymbol{\theta}^{(t)})}{\tilde{\mathcal{L}}(\boldsymbol{\theta}^{(t-1)}; \mathbf{m}, \mathbf{s})\ q(\boldsymbol{\theta}^{(t)}|\boldsymbol{\theta}^{(t-1)})\ f(\boldsymbol{\theta}^{(t-1)})}$$

where $f(\boldsymbol{\theta})$ is a prior density ascribed to the model parameters.

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Since non-fuzzy realizations **y** are thought as unobserved random quantities, in this case the likelihood function is as follows [4]:

$$\tilde{\mathcal{L}}(\boldsymbol{\theta}^{(t)};\mathbf{m},\mathbf{s}) = \prod_{i=1}^{n} \int_0^1 \xi_{\widetilde{y}_i}(y;m_i,s_i)\frac{\Gamma(\phi_i)y^{(\mu_i\phi_i-1)}(1-y)^{(\phi_i-\mu_i\phi_i-1)}}{\Gamma(\phi_i\mu_i)\Gamma(\phi_i-\mu_i\phi_i)}\,dy$$

where

$$\boldsymbol{\xi}_{\tilde{\mathbf{y}}}(\mathbf{y};\mathbf{m},\mathbf{s}) = \prod_{i=1}^{n}\xi_{\widetilde{y}_i}(y;m_i,s_i)$$

follows from the **non-interactive** assumption of fuzzy observations.

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Aim: Investigate predictors of self-report (fuzzy) sexual intimacy [6].

Sample: $n = 450$ participants from Flanders (73% female, mean age 32.9 years, mean relationship length 7.68 years).

Predictors: (i) perceived *partner responsiveness*, (ii) sexual *desire*, (iii) avoidant *attachment score*.

7-point rating scales from 1 ("definitely not") to 7 ("yes, definitely").

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Data analysis: Three additive Beta linear models M1-M3 have been defined to predict sexual intimacy. The models differ in terms of covariates for the term $\boldsymbol{\mu}$ whereas $\boldsymbol{\phi} = \exp(\mathbf{1}\gamma)$.

For all the models, $f(\boldsymbol{\beta}) = \mathcal{N}(\boldsymbol{\beta}; \mathbf{0}, \mathbf{I}10)$ and $f(\gamma) = \mathcal{N}(\gamma; 0, 3)$.

Four parallel MCMCs have been run with 20000 samples and 5000 samples for the burn-in phase.

The final model has been chosen according to the LOO information criterion.

Results

According to the Gelman and Rubin's convergence diagnostics, all the chains reached the convergence (i.e., $\hat{R} = 1.00$).

Models comparison:

Model	Covariates	LOO
M1	`partner_respo`, `sex_desire`	873.80
M2	`partner_respo`, `sex_desire`, `attach_avoid`	863.50
M3	`partner_respo`, `sex_desire`, `attach_avoid`, `gender_partner`	**857.00**

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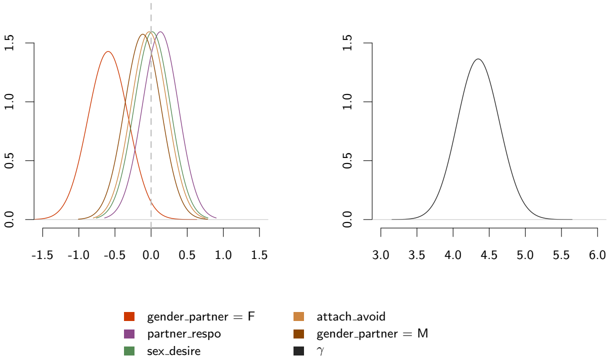
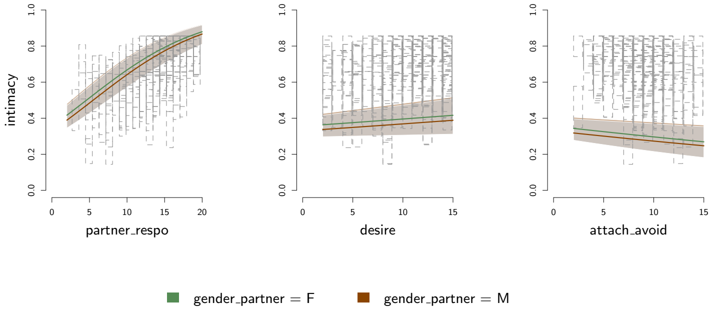
Results

Posterior parameters densities:



Results

Fitted vs. observed fuzzy data (rectangular sections):



Conclusions

- When coupled with standard probability theory, fuzzy numbers can deal with different sources of uncertainty in the same statistical model (i.e., randomness and imprecision).

- With regards to ratings data, fuzzy numbers provide flexible formal representations which might be used to integrate several information from the rating process (e.g., final response, decision uncertainty).

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[1] Calcagnì, A., and Lombardi, L.
Dynamic fuzzy rating tracker (dyfrat): a novel methodology for modeling real-time dynamic cognitive processes in rating scales.
Applied soft computing 24 (2014), 948–961.

[2] Calcagnì, A., Cao, N., Rubaltelli, E., and Lombardi, L.
A psychometric modeling approach to fuzzy rating data.
Fuzzy Sets and Systems (2022), 1–24 (In Press).

[3] de Sáa, S. d. l. R., Gil, M. Á., González-Rodríguez, G., López, M. T., and Lubiano, M. A.
Fuzzy rating scale-based questionnaires and their statistical analysis.
IEEE Transactions on Fuzzy Systems 23, 1 (2014), 111–126.

[4] Gil, M., Corral, N., and Gil, P.
The minimum inaccuracy estimates in $\chi 2$ tests for goodness of fit with fuzzy observations.
Journal of Statistical Planning and Inference 19, 1 (1988), 95–115.

[5] Haario, H., Saksman, E., and Tamminen, J.
An adaptive metropolis algorithm.
Bernoulli (2001), 223–242.

[6] Van Lankveld, J. J., Dewitte, M., Verboon, P., and van Hooren, S. A.
Associations of intimacy, partner responsiveness, and attachment-related emotional needs with sexual desire.
Frontiers in Psychology 12 (2021).

[7] Vonglao, P.
Application of fuzzy logic to improve the likert scale to measure latent variables.
Kasetsart Journal of Social Sciences 38, 3 (2017), 337–344.

[8] Yu, S.-C., and Wu, B.
Fuzzy item response model: a new approach to generate membership function to score psychological measurement.
Quality and Quantity 43, 3 (2009), 381.

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