Approximated Gibbs sampling for continuous fuzzy numbers

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Introduction

Fuzzy data are ubiquitous in many research contexts, including social and behavioral sciences.

Rating data are a typical example of fuzzy data, as the process of measuring human attitudes, motivations, or beliefs involve a certain **degree of uncertainty** and fuzziness.

Fuzzy data are also common in **classification-based problem**, such as when precise data are classified into imprecise categories (e.g., images or scenes classification, content analysis, human-based assessments).

Introduction

In these cases, **statistical models** have to cope with fuzzy data and appropriate methods need to be used in order to make **inference** appropriately.

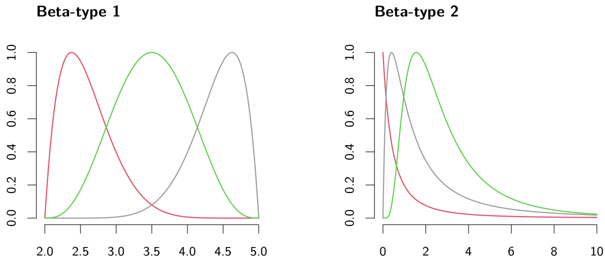
Several methods have been proposed over the years, most of them based on **generalization of likelihood theory** to fuzzy samples [1, 2].

However, statistical **estimators** often suffer from **excessive variance** (i.e., larger standard errors) especially when **epistemic fuzzy data** are considered [3].

Goal:

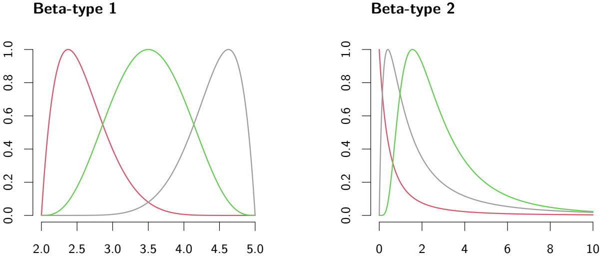
- Define a probabilistic schema to mimic the sampling process underlying epistemic fuzzy data
- Use this mechanism to make inference on the parameters of statistical models
- Plug this mechanism into statistical estimators to reduce variance of estimated parameters

The proposed solution uses Beta-type fuzzy numbers as a general template for representing **continuous** and **unimodal** fuzzy numbers.



A conditional sampling schema

The Beta-type fuzzy numbers



- Flexible and parsimonious as they require **two parameters** only (m: mode; s: precision)
- Deal with variables supported on **bounded** or **semi-infinite** intervals (as those commonly used in social and behavioral research)
- Generalize **triangular** fuzzy numbers as well

A conditional sampling schema

Statement of the problem

Let $Y_1, \ldots, Y_n$ be n independent continuous r.vs. and $\tilde{\mathbf{y}} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ a sample of fuzzy observations. The vector $\tilde{\boldsymbol{y}}$ is a **blurred** version of $\mathbf{y}$ because of *post-sampling* or epistemic uncertainty-based processes.

The interest lies in studying $f_{Y_1,\ldots,Y_n}(\mathbf{y}; \boldsymbol{\theta}_{\boldsymbol{y}})$ with the purpose of making inference on $\boldsymbol{\theta}_{\boldsymbol{y}}$ given the fuzzy sample $\tilde{\boldsymbol{y}}$.

Each fuzzy observation $\tilde{y}_i$ consists of mode and precision $\{m_i, s_i\}$ of a Beta-type fuzzy number.

The idea is to use a **conditional schema** linking the parameters of fuzzy numbers (i.e., mode m and precision s) to $f_{Y_1,\ldots,Y_n}(\mathbf{y};\boldsymbol{\theta}_{\mathbf{y}})$:

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$$
\begin{aligned}
y_i &\sim f_Y(y;\boldsymbol{\theta}_{\boldsymbol{y}}) \\
s_i &\sim f_S(s;\boldsymbol{\theta}_{\boldsymbol{s}}) \\
m_i|y_i, s_i &\sim f_{M|S,Y}(m;\omega(y,s))
\end{aligned}
$$

$$\boxed{y_i \sim f_Y(y; \boldsymbol{\theta_y})}$$
$$s_i \sim f_S(s; \boldsymbol{\theta_s})$$
$$m_i | y_i, s_i \sim f_{M|S,Y}(m; \omega(y, s))$$

Rv governing the stochastic (**non-fuzzy**) sampling process. The parameters can be expressed as a function of external covariates $\boldsymbol{\theta_y} = g^{-1}(\mathbf{X}\beta)$ as for GLMs.

It depends on the specific problem one is dealing with (e.g., Beta distribution, Logistic distribution, Weibull distribution).

$$y_i \sim f_Y(y; \boldsymbol{\theta}_y)$$

$$\boxed{s_i \sim \mathcal{G}a(s; \alpha_s, \beta_s)}$$

$$m_i | y_i, s_i \sim f_{M|S,Y}(m; \omega(y,s))$$

Gamma distribution with $\alpha_s > 0$ and $\beta_s > 0$ modeling the precision (or spread) of the fuzzy number. In the simplest case, $s_i \perp\!\!\!\perp y_i$ although it can be generalized to cope with cases where s_i depends on y_i or external covariates.

$$y_i \sim f_Y(y; \boldsymbol{\theta_y})$$
$$s_i \sim \mathcal{G}a(s; \alpha_s, \beta_s)$$
$$\boxed{m_i | y_i, s_i \sim f_{M|S,Y}(m; \omega(y, s))}$$

Rv for the mode of the fuzzy number as a function of the true unobserved outcome y_i and the spread s_i.

A conditional sampling schema

$$y_i \sim f_Y(y;\boldsymbol{\theta_y})$$
$$s_i \sim \mathcal{G}a(s;\alpha_s,\beta_s)$$
$$\boxed{m_i|y_i,s_i \sim \mathcal{B}e_{4P}(m;s_iy_i,s_i-s_iy_1,lb,ub)}$$

Rv governing the mode of the fuzzy number as a function of the true unobserved outcome y_i and the spread s_i.

Case 1: $y \in (lb,ub) \subset \mathbb{R}$ the **four-parameter Beta distribution** is used.

$$y_i \sim f_Y(y;\boldsymbol{\theta_y})$$
$$s_i \sim \mathcal{G}a(s;\alpha_s,\beta_s)$$
$$\boxed{m_i|y_i,s_i \sim \mathcal{B}e_P(m;y_i+y_is_i,s_i+2)}$$

Rv governing the mode of the fuzzy number as a function of the true unobserved outcome y_i and the fuzziness s_i.

Case 2: $y \in (0,+\infty)$ the **Beta prime distribution** is instead used.

A conditional sampling schema

Proposed solution

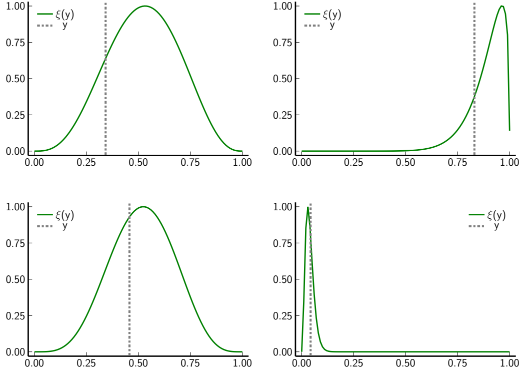
$$y_i \sim f_Y(y; \boldsymbol{\theta_y}) \tag{1}$$

$$s_i \sim \mathcal{G}a(s; \alpha_s, \beta_s) \tag{2}$$

$$m_i | s_i, y_i \sim \begin{cases} \mathcal{B}e_{4P}(m; s_i y_i, s_i - s_i y_1, lb, ub), & \text{if } y_i \in (lb, ub) \\ \mathcal{B}e_P(m; y_i + y_i s_i, s_i + 2), & \text{if } y_i \in (0, +\infty) \end{cases} \tag{3}$$

In both cases, the **fuzziness propagation** through Eq. (3) acts by letting $(m_1, \ldots, m_n)$ spread out near $\mathbb{E}[Y]$.

A conditional sampling schema
Proposed solution



Examples of a Beta-type 1 fuzzy number $\xi_{\widetilde{y}}$ masking the (true) uncorrupted realizations y

Inference about $\boldsymbol{\theta_y}$ involves a kind of **deblurring** procedure which uses $\tilde{\boldsymbol{y}}$ instead of the unobserved realizations $\mathbf{y}$.

Inference on θ_y

Approximated Gibbs sampling

The idea is to plug the hypothesized sampling schema into the estimation procedure, which naturally leads to a **Gibbs sampler**-based solution:

For $t > 1$ do:

$$\mathbf{y}^{(t)} \sim \pi(\mathbf{y}|\mathbf{m}, \mathbf{s}, \boldsymbol{\theta}_y^{(t-1)})$$

$$\boldsymbol{\theta}_{\boldsymbol{y}}^{(t)} \sim \pi(\boldsymbol{\theta}_{\boldsymbol{y}}|\mathbf{m}, \mathbf{s}, \boldsymbol{y}^{(t)})$$

For large T inference on $\boldsymbol{\theta}_{\boldsymbol{y}}$ can be performed by inspection of the posterior sequence $\left(\boldsymbol{\theta}_{\boldsymbol{y}}^{(1)}, \ldots, \boldsymbol{\theta}_{\boldsymbol{y}}^{(T)}\right)$.

Conditional posterior densities $\pi(\mathbf{y}|\ldots)$ and $\pi(\boldsymbol{\theta}_{\mathbf{y}}|\ldots)$ do not have known form under the proposed sampling schema. Then, hybrid solutions, such as **posterior approximation** or **Metropolis within Gibbs** could be used to solve the problem.

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Posterior sampling schema:

$\pi(\boldsymbol{\theta}_{\mathbf{y}}|\mathbf{y},\{\mathbf{m},\mathbf{s}\})$ via **MCMC** [5]

$\pi(\mathbf{y},\boldsymbol{\theta}_{\mathbf{y}}|\{\mathbf{m},\mathbf{s}\})$ $\nearrow$ $\searrow$

$\pi(\mathbf{y}|\boldsymbol{\theta}_{\mathbf{y}},\{\mathbf{m},\mathbf{s}\})$ via quadratic **posterior approximation** [4]

Conditional posterior densities $\pi(\mathbf{y}|\ldots)$ and $\pi(\boldsymbol{\theta}_\mathbf{y}|\ldots)$ do not have known form under the proposed sampling schema. Then, hybrid solutions, such as **posterior approximation** or **Metropolis within Gibbs** could be used to solve the problem.

Posterior sampling schema:

$$\pi(\boldsymbol{\theta}_\mathbf{y}|\mathbf{y},\{\mathbf{m},\mathbf{s}\}) \quad \text{via } \textbf{MCMC} \text{ [5]}$$

$$\pi(\mathbf{y},\boldsymbol{\theta}_\mathbf{y}|\{\mathbf{m},\mathbf{s}\}) \nearrow \searrow$$

$$\boxed{\pi(\mathbf{y}|\boldsymbol{\theta}_\mathbf{y},\{\mathbf{m},\mathbf{s}\})} \quad \text{via quadratic } \textbf{posterior approximation} \text{ [4]}$$

$\boxed{\textbf{Case 1}: y \in (lb, ub)}$

$$\ln \pi(y_i|\boldsymbol{\theta}_{\mathbf{y}}, \ldots) \propto -\ln \Gamma(y_i^* s_i) - \ln \Gamma(s_i - s_i y_i^*) + s_i y_i^* \ln\left(\frac{m_i - lb}{ub - m_i}\right) \;+\; \ln f_Y(y; \boldsymbol{\theta}_{\mathbf{y}})$$

$$\stackrel{\sim}{=} \ln \mathcal{B}e_{4P}(y; \lambda\sigma, \sigma - \sigma\lambda, lb, ub)$$

$y_i^* = (y_i - lb)/(ub - lb)$

$\boxed{\textbf{Case 1} : y \in (lb, ub)}$

$$\ln \pi(y_i|\boldsymbol{\theta_y},\ldots) \propto \overbrace{-\ln\Gamma(y_i^* s_i) - \ln\Gamma(s_i - s_i y_i^*) + s_i y_i^* \ln\left(\frac{m_i - lb}{ub - m_i}\right)}^{h(y;m,s,lb,ub)} + \ln f_Y(y;\boldsymbol{\theta_y})$$
$$\stackrel{\sim}{=} \ln \mathcal{B}e_{4P}(y; \lambda\sigma, \sigma - \sigma\lambda, lb, ub)$$

$\{\lambda, \sigma\} \in (lb, ub) \times \mathbb{R}^+$:

$$\frac{\partial^k}{\partial y^k} \ln \mathcal{B}eta_{4P}(y; \lambda\sigma, \sigma - \sigma\lambda, lb, ub) = \frac{\partial^k}{\partial y^k}\Big(h(y; m, s, lb, ub) + \ln f_Y(y; \boldsymbol{\theta_y})\Big)$$
$$k = 1, 2$$

$$\boxed{\textbf{Case 2}: y \in (lb, +\infty)}$$

$$\begin{aligned}\ln \pi(y_i|\boldsymbol{\theta_y},\ldots) &\propto \ln \mathrm{B}(y_i+s_i, s_i+2)^{-1} + \ln\left(\frac{m_i}{m_i+1}\right)(y_i + s_i y_i) + \ln m_i + 2\ln(1+m_i) + \\ &+ \ln f_Y(y;\boldsymbol{\theta_y}) \\ &\stackrel{\sim}{=} \ln \mathcal{B}e_P(y;\lambda+\lambda\sigma, \sigma+2)\end{aligned}$$

$\boxed{\textbf{Case 2} : y \in (lb, +\infty)}$

$$\ln \pi(y_i|\boldsymbol{\theta_y},\ldots) \propto \overbrace{\ln \mathrm{B}(y_i + s_i, s_i + 2)^{-1} + \ln\left(\frac{m_i}{m_i+1}\right)(y_i + s_i y_i) + \ln m_i + 2\ln(1+m_i)}^{g(y;m,s)} +$$

$$+ \ln f_Y(y; \boldsymbol{\theta_y})$$

$$\stackrel{\sim}{=} \ln \mathcal{B}e_P(y; \lambda + \lambda\sigma, \sigma + 2)$$

$$\{\lambda, \sigma\} \in (0, +\infty) \times \mathbb{R}^+ :$$

$$\frac{\partial^k}{\partial y^k} \ln Be_P(y; \lambda + \lambda\sigma, \sigma + 2) = \frac{\partial^k}{\partial y^k}\Big(g(y; m, s) + \ln f_Y(y; \boldsymbol{\theta_y})\Big)$$

$$k = 1, 2$$

Aim: Assessing the quadratic posterior approximation of $\pi(y|\boldsymbol{\theta}_{\boldsymbol{y}},\ldots)$ via $\mathcal{B}e_{4P}$ and $\mathcal{B}e_P$ distributions for both bounded and left-bounded cases.

Methods: The derivative-based density approximation (DA) is contrasted against the Adaptive Rejection Sampling (ARS) algorithm.

Measures:

- Total variation distance: $d_{TV} = \frac{1}{2}\int \left|\tilde{\pi}(y|\ldots) - \pi(y|\ldots)\right| dy$
- Computation time

$\boxed{\textbf{Case 1} : y \in (lb, ub)}$

$f_Y(y; \boldsymbol{\theta_y}) \stackrel{\text{def}}{=} \mathcal{LG}Norm(y; \mu, \phi)$ *Logit Normal distribution*

$lb = 0, ub = 1$

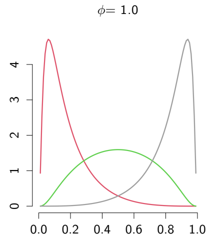
$\mu \in \{-1.85, 0, 1.85\}$

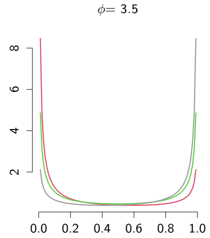
$\phi \in \{1.0, 3.5\}$

$s \sim \mathcal{G}a(s; 45.0, 45.0/\mu_s)$

$\mu_s \in \{5.0, 25.0, 50.0\}$

$n = 2000$ replicates for each combination



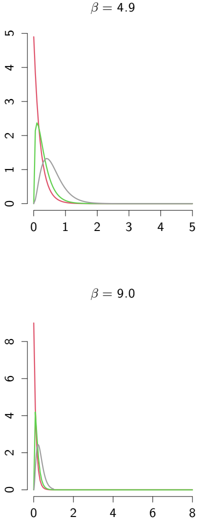


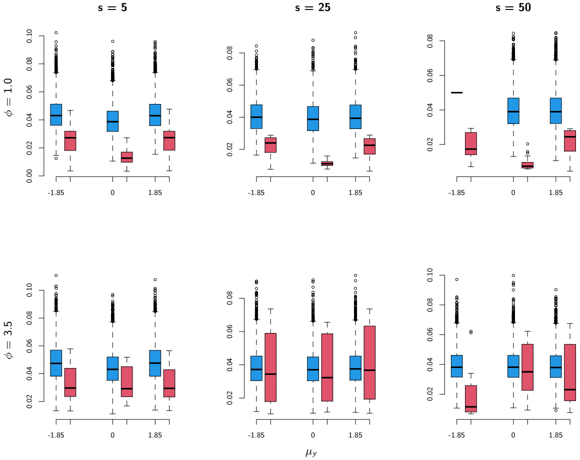
$$\boxed{\textbf{Case 2} : y \in (0, +\infty)}$$

$f_Y(y; \boldsymbol{\theta_y}) \stackrel{\text{def}}{=} \mathcal{G}a(y; \alpha, \beta)$ *Gamma distribution*
$\alpha \in \{1.0, 1.5, 3.0\}$
$\beta \in \{4.9.0, 9.0\}$

$s \sim \mathcal{G}a(s; 45.0, 45.0/\mu_s)$
$\mu_s \in \{5.0, 25.0, 50.0\}$

$n = 2000$ replicates for each combination

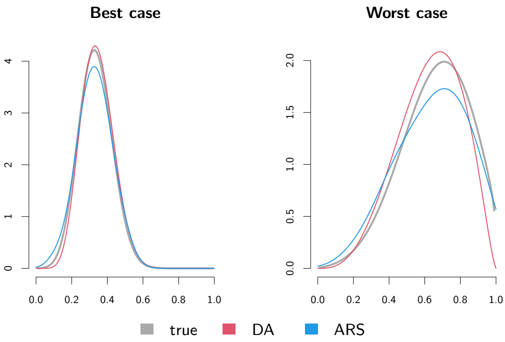




	DA	ARS
Average Accuracy	0.97	0.95
Average Log-Time	-6.97	0.67

Simulation study

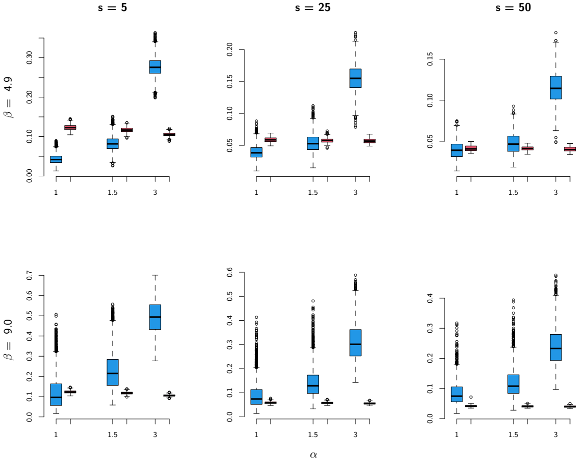
Results - Case 1



	DA	ARS
Average Accuracy	0.97	0.95
Average Log-Time	-6.97	0.67

Simulation study

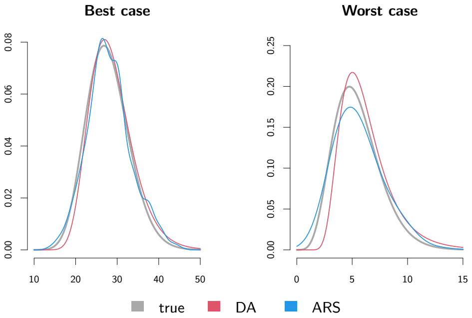
Results - Case 2



	DA	ARS
Average Accuracy	0.93	0.85
Average Log-Time	-7.27	1.15

Simulation study

Results - Case 2



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Average Accuracy	0.93	0.85
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Conclusions

- A general framework for **data analysis** by taking the advantages of a **general sampling schema** for the fuzziness propagation over the outcomes of $f_Y(y; \boldsymbol{\theta_y})$
- Results are still **preliminary**: Further simulations coupling DA with MCMC are currently underway
- Generalizations to non-convex and **trapezoidal fuzzy numbers** need also to be considered

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